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Linear Quadratic Optimal Control of an Inverted Pendulum on a Cart using Artificial Bee Colony Algorithm: An Experimental Study

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Abstract

This study presents a Linear Quadratic Optimal (LQR) controller design for an inverted pendulum on a cart using the Artificial Bee Colony (ABC) algorithm. Main design parameters of the linear quadratic regulator are the weighting matrices. Generally, selecting weighting matrices is managed by trial and error since there exists no apparent connection between these weights and time domain requirements such as settling time, steady state error, and overshoot percentage. In this study after deriving the mathematical models of the inverted pendulum on a cart and the DC motor, an LQR controller is designed using the ABC algorithm to determine weighting matrices to overcome LQR design difficulties. The comparison and experimental results justify that the ABC algorithm is a very efficient way to determine LQR weighting matrices in comparison with a method in literature.

Keywords: Artificial bee colony algorithm, Linear quadratic regulator, Inverted pendulum

Yapay Arı Kolonisi Algoritması ile Bir Arabalı Ters Sarkacın Lineer Kuadratik Kontrolü: Deneysel Bir Çalışma

Öz

Bu çalışmada, Lineer Kuadratik Regülatör (LQR) ile bir ters sarkacın kontrolü için, Yapay Arı Kolonisi (ABC) optimizasyon algoritmasına dayalı bir metot önerilmiştir. LQR'ın temel tasarım parametreleri ağırlık matrisleridir. Ağırlık matrislerinin değerleri ile yüzde aşımı, yerleşme zamanı ve kararlı hal hatası gibi zaman uzayı performans kriterleri arasında doğrudan bir ilişki olmadığı için bu matrislerin seçimi genellikle deneme yanılma yöntemiyle gerçekleştirilmektedir. Bu çalışmada arabalı ters sarkaç ve bu mekanizmayı hareket ettiren DC motorun matematiksel modellerinin elde edilmesinin ardından sürü zekası temelli bir optimizasyon algoritması olan ABC algoritması kullanılarak bir LQR kontrolör tasarlanmıştır. Karşılaştırma ve deney sonuçları, ABC algoritmasının literatürde önerilen bir yöntem ile karşılaştırıldığında ağırlık matrislerinin belirlenmesinde daha etkin bir yol olduğunu göstermiştir.

Anahtar Kelimeler: Yapay arı kolonisi algoritması, Lineer kuadratik regülatör, Ters sarkaç

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1. INTRODUCTION

Controlling an inverted pendulum on a cart is a challenging problem due to the various characters of the system: including highly unstable, nonlinear, non-minimum phase, underactuated, and single input-two output mechanical system. Several substantial control systems can be modelled with the help of inverted pendulum [1]. Inverted pendulum reveals many interesting system - theoretic properties and its dynamics are fundamental for balancing problems [2,3].

The inverted pendulum system set up consists of a cart, a pendulum, a DC motor, and a driving mechanism. There are two equilibrium points being stable (downwards position) and unstable (upwards position) for this system. Hence, there are two control objectives for the inverted pendulum system. First one is to swing the pendulum up to unstable equilibrium from stable equilibrium and the second one is to maintain the unstable equilibrium position. This study will focus on the design of an optimal controller for the second control objective for the inverted pendulum system. During this study Feedback Instruments' digital pendulum system [4] is used to create a more realistic control system.

One of the widely used optimal control techniques is the Linear Quadratic Regulator (LQR) [5, 6]. The challenging part of the LQR controller design is to choose its weights for states and control signal [7]. Generally selecting these matrices is based on the knowledge of control engineers or designers and this process takes a long time since it is carried out by trial and error. Various methods to determine suitable weighting matrices have been proposed [5, 8]. One of the methods for choosing weighting matrices for LQR is Bryson's rules [9]. According to this method Q can be taken as $Q=C^T\bar{Q}C$. Since matrix C includes important outputs, these states are included within the cost. Also another method for choosing weighting matrices for linear quadratic control of an inverted pendulum is proposed by Ghosh et al. [10]. According to this method, the Q matrix can be chosen as $Q = diag\{q_{11}, q_{22}, q_{33}, q_{44}\}$ and $R=r_{11}$ where $q_{11}=500q$, q_{22},$q_{33}=20q$, $q_{44}=q$ and $r_{11}=10^n$. q and n should be estimated by trial and error [10].

Beside these methods, recently many researchers have proposed artificial intelligence algorithms such as Genetic Algorithms (GA) and Particle Swarm Optimization (PSO) algorithm for this goal [11,12]. In addition, there is another computational intelligence algorithm referred to as the Artificial Bee Colony (ABC) algorithm for updating the LQR weights. A discrete-time LQR controller using the ABC algorithm is designed based on only simulation [13]. However, real implementation is very important to show efficiency of the method. The ABC algorithm was introduced by Karaboga [14,15] as a novel method. It is a fast converging algorithm and has only a few parameters to be adjusted. Because of these superiorities it is utilized to find a solution for many higher dimensional linear or nonlinear problems [15,16].

This study proposes a method that selects appropriate weighting matrices for an LQR controller to stabilize cart position and pole angle of a nonlinear inverted pendulum system with DC motor while minimizing settling time, steady state error and overshoot percentage of the output signal as position of the cart. The ABC algorithm is employed to determine weighting matrices.

2. INVERTED PENDULUM

Generally, an inverted pendulum system is composed of a cart and a rod hinged it. The cart is moved by a DC motor. The DC motor supplies some force needed for motion of the cart via a pulley-belt mechanism. Dynamics of the inverted pendulum system can be represented as a set of equations which is called mathematical model. Either this model can be represented in transfer function form or state space form. In this section the complete mathematical model of the inverted pendulum system has been derived.

The parametric representation of the inverted pendulum system is shown in Figure 1 and the parameters are presented in Table 1.

Table 1. Parameters of the inverted pendulum

Variable	Meaning	Unit
x	Displacement of cart	m
θ	Pendulum angle	rad
M	Mass of cart	kg
m	Mass of pendulum	kg
l	Length of pendulum	m
g	Acceleration of gravity	m/s^2
J_p	Moment of inertia	kgm^2
d	Damping coefficient	Nms/rad
b	Friction coefficient	Ns/m

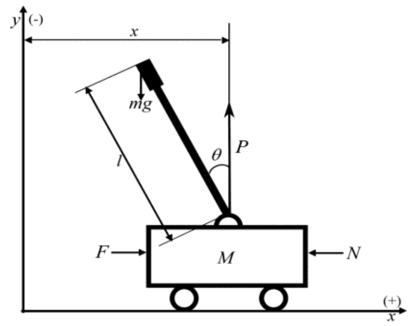


Figure 1. Representation of inverted pendulum

Let N and P be horizontal and vertical components of the force as shown in Figure 1. Considering Figure 1, we can compute the coordinates of center of gravity of the mass:

$$x_G = x + l_x = x - lsin(\theta) \quad (1)$$

$$y_G = l_y = lcos(\theta) \quad (2)$$

According to the Newton's First Law of Motion, applied force on the cart equals the product of mass and its acceleration.

$$F = ma \quad (3)$$

So, the horizontal reaction force becomes:

$$N = m\frac{d^2}{dt^2}(x - lsin(\theta)) \quad (4)$$

Noting $d^2/dt^2\,[\sin(\theta)] = \cos(\theta)\,\ddot{\theta} - \sin(\theta)\dot{\theta}^2$ and $d^2/dt^2\,[\cos(\theta)] = -\sin(\theta)\ddot{\theta} - \cos(\theta)\,\dot{\theta}^2$, Equation 4 can be rewritten as

$$N = m(\ddot{x} - \ddot{\theta}lcos(\theta) + \dot{\theta}^2 lsin(\theta)) \quad (5)$$

The force F applied on the cart is equal to the sum of the forces due to friction, acceleration, and the horizontal reaction

$$F = M\ddot{x} + b\dot{x} + N \quad (6)$$

Substituting Equation 5 in Equation 6 we can get

$$F = (M + m)\ddot{x} + b\dot{x} - ml\ddot{\theta}cos(\theta) + ml\dot{\theta}^2 sin(\theta) \quad (7)$$

To obtain the second equation of motion for the inverted pendulum, we need to add up the forces perpendicular to the pendulum. Considering Figure 1 vertical force P is calculated via the weight of the pendulum. Let $y_G = lcos(\theta)$ be the displacement of pendulum from the pivot. Thus, P is given by

$$P - mg = m\frac{d^2}{dt^2}(lcos(\theta)) \quad (8)$$

Equation 8 can be written as

$$P = -ml\ddot{\theta}sin(\theta) - ml\dot{\theta}^2 cos(\theta) + mg \quad (9)$$

Noting that the torque equation is $\vec{\tau} = \vec{l} \otimes \vec{F} = \begin{bmatrix} \vec{x} & \vec{y} & \vec{z} \\ l_x & l_y & 0 \\ -N & -P & 0 \end{bmatrix}$ where the notation $\otimes$ indicates vector product

$$\tau = Plsin(\theta) + Nlcos(\theta) \quad (10)$$

And also

$$\tau = J_p\ddot{\theta} + d\dot{\theta} \quad (11)$$

where J_p is pendulum moment of inertia and d is pendulum damping coefficient. Equating Equation 10 and Equation 11, we can get

$$Plsin(\theta) + Nlcos(\theta) = J_p\ddot{\theta} + d\dot{\theta} \quad (12)$$

Using the well-known trigonometric equation $\cos^2(\theta)+\sin^2(\theta)=1$, Equation 12 can be rewritten as

$$(J_p+ml^2)\ddot{\theta}-mglsin(\theta)-ml\ddot{x}cos(\theta)+d\dot{\theta}=0 \quad (13)$$

Equation 7 and Equation 13 are the equations of motion for the inverted pendulum that describe the translational and rotational motion, respectively. From Equation 7 and Equation 13, $\ddot{x}$ and $\ddot{\theta}$ can respectively be written as

$$\begin{aligned}\ddot{x}=&\frac{-(J_p+ml^2)b\dot{x}+m^2l^2gcos(\theta)sin(\theta)}{\sigma}\\&+\frac{-mlcos(\theta)d\dot{\theta}-(J_p+ml^2)ml\dot{\theta}^2sin(\theta)}{\sigma}\\&+\frac{(J_p+ml^2)F}{\sigma}\end{aligned} \quad (14)$$

$$\begin{aligned}\ddot{\theta}=&\frac{(M+m)mglsin(\theta)-mlbcos(\theta)\dot{x}}{\sigma}\\&+\frac{-m^2l^2cos(\theta)sin(\theta)\dot{\theta}^2-(M+m)d\dot{\theta}}{\sigma}\\&+\frac{mlcos(\theta)F}{\sigma}\end{aligned} \quad (15)$$

where $\sigma=(J_p+ml^2)(M+m)-m^2l^2cos^2(\theta)$.

3. INVERTED PENDULUM WITH DC MOTOR

In the inverted pendulum system, the cart is driven by a DC motor. To create a more realistic model, the motor characteristics should be added to the mathematical model of the inverted pendulum system. In this section the mathematical model of the DC motor has been analyzed and then it has been applied to the inverted pendulum model.

The transfer function is derived for an armature-controlled DC [17,18]. The voltage is called back electromotive force which is proportional to speed:

$$v_b(t)=K_b\frac{d\theta_m(t)}{dt} \quad (16)$$

where K_b is the back electromotive force constant and $d\theta_m(t)/dt=\omega(t)$ is the angular velocity of the motor. Taking the Laplace transform of Equation 16 gives

$$V_b(s)=K_bs\theta_m(s) \quad (17)$$

The relation $v_b(t)$ between the armature current $i(t)$ and the armature voltage $e(t)$ can be written in Laplace form as

$$RI(s)+LsI(s)+V_b(s)=E(s) \quad (18)$$

where R and L are rotor circuit resistance and rotor circuit inductance, respectively. The torque generated by the DC motor is proportional to the armature current:

$$T_m(s)=K_tI(s) \quad (19)$$

where T_m is the torque, and K_t is the torque constant. Rearranging Equation 19 for $I(s)$ and substituting it and also substituting Equation 17 into Equation 18 yields

$$\frac{(R+Ls)T_m(s)}{K_t}+K_bs\theta_m(s)=E(s) \quad (20)$$

The torque developed by the motor also can be written as follows:

$$T_m(s)=(J_ms^2+Ds)\theta_m(s) \quad (21)$$

where J_m is the inertia of the motor and, D is the viscous damping. Substituting Equation 21 into Equation 20 yields

$$\frac{\theta_m(s)}{E(s)}=\frac{K_t}{(R+Ls)(J_ms^2+Ds)+K_bK_ts} \quad (22)$$

Equation 22 is the transfer function of the DC motor between input (voltage) and output (angular position). Noting that $\omega(s)=s\theta(s)$ and substituting $\omega(s)/s$ instead of $\theta(s)$ into Equations. 22 and 20 yields, respectively:

$$\frac{\omega(s)}{E(s)}=\frac{K_t}{(R+Ls)(J_m s+D)+K_b K_t} \tag{23}$$

$$\frac{(R+Ls)T_m(s)}{K_t}+K_b\omega(s)=E(s) \tag{24}$$

In order to obtain torque that is developed by the motor and to get rid of angular velocity, we substitute $\omega(s)$ from Equation 23 into Equation 24, additionally writing $T_m(s)$ term on the left-hand-side yields

$$T_m(s)=\frac{K_t}{R+Ls}\left(1-\frac{K_b K_t}{(R+Ls)(J_m s+D)+K_b K_t}\right)E(s) \tag{25}$$

Equation 25 is the motor torque equation without angular velocity in the equation. Now let us obtain the force equation induced by the motor torque:

$$T_m=F\frac{\rho}{n_1} \tag{26}$$

where ρ and n_1 are radius of pulley and gear ratio, respectively. Substituting Equation 26 into Equation 25 we obtain

$$\begin{aligned}F(s)&=\left(\frac{n_1}{\rho}\right)\frac{K_t}{R+Ls}\\&\times\left(1-\frac{K_b K_t}{(R+Ls)(J_m s+D)+K_b K_t}\right)E(s)\end{aligned} \tag{27}$$

Instead of the force F in the inverted pendulum equations of motion let us use the DC motor armature voltage $E(s)$ as input. Up to this end, let us rearrange Equation 27 considering Equation 23 and translational velocity - angular velocity equation $\omega(s)=-(n_2/\rho)sx(s)$ we obtain

$$F(s)=\left(\frac{n_1}{\rho}\right)\left(-\left(\frac{n_2}{\rho}\right)\frac{K_b K_t}{R+Ls}sx(s)+\frac{K_t}{R+Ls}E(s)\right) \tag{28}$$

where n_2 is gear ratio.

For simplification substituting $L=0$ in Equation 28 and taking the inverse Laplace transform, we can obtain a differential equation whose inputs are motor armature voltage $e(t)$ and translational velocity of the cart $\dot{x}(t)$, and output is the force $F(t)$ applied on the cart.

$$F(t)=-\left(\frac{n_1}{\rho}\right)\left(\frac{n_2}{\rho}\right)\frac{K_b K_t}{R}\dot{x}(t)+\left(\frac{n_1}{\rho}\right)\frac{K_t}{R}e(t) \tag{29}$$

Let the states be $[x_1\ x_2\ x_3\ x_4]=[x\ \dot{x}\ \theta\ \dot{\theta}]$. Using Equations. 14, 15, and 29, the state equations of the inverted pendulum which include voltage e as input, can be written as follows:

$$\dot{x}_1=x_2 \tag{30}$$

$$\begin{aligned}\dot{x}_2=&\frac{-(J_p+ml^2)\left(b+\left(\frac{n_1}{\rho}\right)\left(\frac{n_2}{\rho}\right)\frac{K_b K_t}{R}\right)x_2}{\sigma}\\&+\frac{-m^2l^2gcos(x_3)sin(x_3)-mlcos(x_3)dx_4}{\sigma}\\&+\frac{-(J_p+ml^2)mlx_4^2sin(x_3)(J_p+ml^2)}{\sigma}\\&\times\frac{\left(\left(\frac{n_1}{\rho}\right)\frac{K_t}{R}\right)e}{\sigma}\end{aligned} \tag{31}$$

$$\dot{x}_3=x_4 \tag{32}$$

$$\begin{aligned}\dot{x}_4=&\frac{(M+m)mglsin(x_3)}{\sigma}\\&+\frac{-mlcos(x_3)\left(b+\left(\frac{n_1}{\rho}\right)\left(\frac{n_2}{\rho}\right)\frac{K_b K_t}{R}\right)x_2}{\sigma}\\&+\frac{-m^2l^2cos(x_3)sin(x_3)x_4^2-(M+m)dx_4}{\sigma}\\&+\frac{mlcos(x_3)\left(\left(\frac{n_1}{\rho}\right)\frac{K_t}{R}\right)e}{\sigma}\end{aligned} \tag{33}$$

where $\sigma=(J_p+ml^2)(M+m)-m^2l^2cos^2(x_3)$.

Since the main aim of this study is to design a controller to stabilize the pendulum, so as to retain pendulum upright position in response to changes in cart position, linearization of the equations about the vertically upward equilibrium position, $\theta = 0$, is needed. So, assume very small deviation θ from equilibrium: $\sin(\theta) = \theta, \cos(\theta) = 1$ and $\dot{\theta}^2 = 0$.

Linearization of Equations 30-33 about $\theta = 0$ can be carried out as follows:

$$\dot{x}_1 = x_2 \tag{34}$$

$$\dot{x}_2 = \frac{-(J_p + ml^2)\left(b + \left(\frac{n_1}{\rho}\right)\left(\frac{n_2}{\rho}\right)\frac{K_b K_t}{R}\right)x_2}{\bar{\sigma}} + \frac{m^2 l^2 g x_3 - mld x_4 + (J_p + ml^2)}{\bar{\sigma}} \times \frac{\left(\left(\frac{n_1}{\rho}\right)\frac{K_t}{R}\right)e}{\bar{\sigma}} \tag{35}$$

$$\dot{x}_3 = x_4 \tag{36}$$

$$\dot{x}_4 = \frac{(M+m)mglx_3}{\bar{\sigma}} - \frac{ml\left(b + \left(\frac{n_1}{\rho}\right)\left(\frac{n_2}{\rho}\right)\frac{K_b K_t}{R}\right)x_2}{\bar{\sigma}} - \frac{(M+m)dx_4 + ml\left(\left(\frac{n_1}{\rho}\right)\frac{K_t}{R}\right)e}{\bar{\sigma}} \tag{37}$$

where $\bar{\sigma} = (J_p + ml^2)(M+m) - m^2 l^2$

4. LINEAR QUADRATIC REGULATOR

A linear time-invariant (LTI) and continuous time control system in the state space is represented as [19]

$$\dot{x} = Ax + Bu \qquad y = Cx \tag{38}$$

where x and u are state and control vectors, respectively. A, B, and, C are the matrices of the system.

Assuming that all states are measured or observed and also they are controllable we can determine a gain matrix K for feedback. Using this matrix and the states the control signal is obtained by

$$u(t) = -Kx(t) \tag{39}$$

In the LQR design, in order to calculate optimal control signal a quadratic performance function is minimized:

$$J = \int_0^\infty (x^T Q x + u^T R u)dt \tag{40}$$

where Q and R are positive definite real symmetric matrices. The superscript (T) points out a matrix transpose.

Noting that the term u^Tgiven in Equation 40 represents the expenditure of the energy of control signals. The weighting matrices Q and R play a central role upon making a decision on which one of the two terms in Equation 40 is more important. In order that the control signal $u(t) = -Kx(t)$ makes the dynamic system given in Equation 38 optimal for any initial state $x(0)$ we can determine the matrix K minimizing the performance index given in Equation 40) [19].

$$K = R^{-1} B^T P \tag{41}$$

$$A^T P + PA - PBR^{-1} B^T P + Q = 0 \tag{42}$$

5. ARTIFICIAL BEE COLONY ALGORITHM

The Artificial Bee Colony (ABC) algorithm is a swarm intelligence method to optimize numerical functions [14]. The algorithm uses finding food source ability of honey bee colonies as a model to solve optimization problems [14,15]. In the ABC algorithm, the colony contains three groups of bees: employed, onlooker, and scout. An employed bee goes to a food source to harvest the sweet secretion of plant called nectar and provides knowledge about the food source. An onlooker bee gets the knowledge about food sources from employed bees and chooses a food source. A scout bee seeks for novel food sources randomly. Every food source corresponds to only one employed bee in the colony. So the number of the employed bees is equal to the food sources around the hive. Also the number of the employed bees is equal to the number of the onlooker bees. If an employed bee abandons its food source, it becomes a scout bee [20].

At the beginning of the process, bees select a set of food source positions randomly and determine the nectar amounts of the selected food sources in the ABC algorithm. Then, these bees return the hive and share the information with the other bees, waiting on the dance area. After sharing the information employed bees return to the food source which selected by themselves. If an employed bee consumes the food source, it starts to look for another source in the neighbourhood of the previous one. Then, an onlooker bee chooses a food source depending on the knowledge provided by the employed bees. This division of labour between onlooker bees and employed bees provide the exploitation of local sources and scout bees provide the exploration of new sources [20].

The position of a food source is a possible solution of the optimization problem in the ABC algorithm. The quality of the solution depends on the nectar amount of the associated food source. Initial population of food source positions (SN) is created randomly by the ABC algorithm at the first step. Each food source as a solution $x_i(i = 1,2, \ldots, SN)$ is a D_{op} -dimensional vector where D_{op} denotes the number of the parameters in the optimization problem [20].

After the initialization of the ABC algorithm, all bees search every food source until a predetermined number of iterations *Cycle=1,2,…,MCN*, where MCN represents the maximum cycle number. An employed bee generates some changes on the food source for encountering a new food source and checks the nectar amount of new food source. If the new nectar amount is more abundant than the previous one, the position of new sources is replaced with the old position. Otherwise, the bee keeps the position of the previous food source. After all employed bees finish their search process, the nectar information and positions of the food sources are shared with the onlooker bees. An onlooker bee evaluates the information of nectar amounts of food sources from employed bees then makes a choice for a food source by using a selection probability with respect to that evaluated information [20].

An onlooker bee selects a food source depending on the probability value of that food source, p_i, determined by the Equation 43) as follows [20]:

$$p_i = \frac{fit_i}{\sum_{i=1}^{SN} fit_i} \tag{43}$$

where fit_i denotes the fitness value of the solution i and it is proportional to the nectar amount of the food source in position i.

The ABC algorithm utilizes the Equation 44 to create a new food position from the old position in the memory [20].

$$v_{ij} = x_{ij} + \phi_{ij}(x_{ij} - x_{kj}) \tag{44}$$

where $j \in 1,2, \ldots, D_{op}$ and $k \in 1,2, \ldots, SN$ are randomly selected indexes. Despite it is determined randomly, the index k should be different from i. ϕ_{ij} is determined in the range of $[-1,1]$ randomly, and controls the production of a food source position around the neighbourhood of x_{ij}. As seen from Equation 44, as long as the difference between the parameters x_{ij} and x_{kj} decreases, the perturbation on the position x_{ij}

decreases. Hence, as the search comes close to optimum solution in the search space, the step length is adaptively reduced.

The food source left by the employed bees is replaced with a new food source by scout bees. In the ABC algorithm, this is performed by producing a position randomly and changing it with the previous one. If a position cannot become better further during a predetermined number of cycles which is called $limit$, then that food source is thought to be abandoned. After the abandonment of a food source, scout bees discover a new food source to replace it. This process can be defined as follows [20]:

$$x_{ij} = x_{jmin} + rand[0,1](x_{jmax} - x_{jmin}) \tag{45}$$

where $j \in 1,2,...,D_{op}$ and $i \in 1,2,...,SN$.

If the nectar amount of new source is equal or better than the nectar amount of the old source, the new source position is changed with the old one or else the old food position is kept in the memory. So, a greedy selection mechanism is engaged as the selection operation between the candidate food source and the old one [20].

6. LQR CONTROLLER DESIGN USING THE ABC ALGORITHM

Determination of the matrices Q and R of the LQR weights considerably affects the performance of the controlled system. In a general manner determining the weights is carried out by trial and error as mentioned before.

In this study the ABC algorithm is employed to select Q and R to design an LQR controller for considering both pendulum's angle and cart's position. In this sections we shall explain how to determine the matrices Q and R of the LQR using the ABC algorithm in order to take time-domain specifications into account.

Main design parameters of the LQR are weighting matrices. Computing these weighting matrices to minimize a performance index by the ABC algorithm is a minimization problem. This problem requires to be resolved in such a way that output of the system attains the desired level in the shortest time as far as possible without a high overshoot. Hence, the objective of this design is to decrease the settling time $t(s)$ of the output of the system (the cart position) with no overshoot (os) or with minimum overshoot and also minimize steady-state error (e_{ss}). The objective weighting method where multiple objective functions are united in one objective function f_{sum} can be employed for multiple objective optimization [21]. The objective function f_{sum} can be written as

$$f_{sum} = K_1 t_s + K_2 os + K_3 e_{ss} \tag{46}$$

where K_1, K_2 and K_3 are the coefficients of the fitness functions and their values are selected as 1.0 by trial and error in this study.

In addition to time-domain specifications included in the objective function given in Equation 46, important physical constraints of the controlled system should be added to the objective function. The inverted pendulum system has two physical constraints. The first one is the bound of the control signal which must be in the range of -2.5V and +2.5V and the second one is the cart position which is physically bounded by the rail length which is 1m. Since, it is assumed that the initial cart position is in the middle of the rail, position of the cart can be limited to $|x| \leq 0.5$ [4].

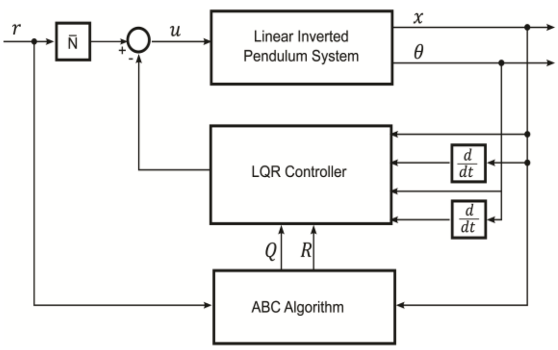


Figure 2. Block diagram of the ABC training

The cart position constraint is already incorporated into the objective function f_{sum} which is given in Equation 46, through the overshoot, on the other hand, to incorporate the control signal u into the ABC algorithm fitness function, a penalty method is used. The objective function given in Equation 46 can be evaluated with this method in the following way [21]:

$$\text{evaluate } f(x) = \begin{cases} f_{sum}, & \text{if } |u| \leq 2.5 \\ f_{sum} + \Psi, & \text{otherwise} \end{cases} \quad (47)$$

where ψ denotes a penalty coefficient for violation of the control signal bound. The penalty coefficient ψ is 0 if $|u| \leq 2.5$ otherwise it is a positive constant. In this study ψ is selected as 5 by trial and error.

Block diagram of the ABC training is shown in Figure 2. The reference input is selected as $r = 0.1$. A pre-compensation scale factor $\bar{N}$ which addresses the steady state error, is added to the reference input as shown in Figure 2. Pre-compensation scale factor $\bar{N}$ can be defined as follows [22]:

$$\bar{N} = -\left(C_n (A - BK)^{-1} B\right)^{-1} \quad (48)$$

where $C_n = [1\ 0\ 0\ 0]$ to ensure that the reference input will be only applied to the first state which is the position of the cart.

As mentioned in the previous sections, Q and R weighting matrices must be real symmetric and positive definite matrices to ensure stability. Also Q and R matrices must be non-negative definite to ensure $x^T Q x$ and $u^T R u$ non-negative for all possible x and u in Equation 40. The easiest way to ensure that the matrices are non-negative definite is picking the weighting matrices to be diagonal with all diagonal elements positive or zero [23]. Selecting diagonal weighting matrices causes the interaction of the components of the states and control to decrease. However, this is not a unique way to guarantee the weighting matrices to be positive definite. There exists another way in matrix theory to make sure that a matrix Q is positive definite. Noting the algebraic principle; a real symmetric matrix Q is positive definite if there exists a non-singular matrix $\bar{Q}$ such that [24]

$$Q = \bar{Q}\bar{Q}^T \quad (49)$$

Since Q matrix is defined in the form $Q = \bar{Q}\bar{Q}^T$ and R matrix $R = \bar{R}\bar{R}^T$, Q and R would be positive definite in any case. In this way we have more degree of freedom to select elements of matrices than diagonal matrices. Because there is not a direct relation with the matrices' elements and performance specifications, choosing weighting matrices is not simple. In order to meet these constraints the ABC algorithm was employed to select the best Q and R matrices that minimize the Equation 46 in view of the time-domain performance specifications.

The main parameters of the ABC algorithm, colony size and maximum number of cycles are set as SN=20 and MCN=100 by trial and error. To create two matrices which $\bar{Q}$ is 4×4 and $\bar{R}$ is 1×1 non-singular real symmetric matrices, the number of parameters of the problem to be optimized is defined as 11 and they will be denoted as $X_1, X_2, \ldots, X_{11}$. Also the lower and upper bounds of the parameters are set 0.1 and 20, respectively, by trial and error. $\bar{Q}$ and $\bar{R}$ matrices are defined as follows

$$\bar{Q} = \begin{bmatrix} \bar{q}_{11} & \bar{q}_{12} & \bar{q}_{13} & \bar{q}_{14} \\ \bar{q}_{21} & \bar{q}_{22} & \bar{q}_{23} & \bar{q}_{24} \\ \bar{q}_{31} & \bar{q}_{32} & \bar{q}_{32} & \bar{q}_{34} \\ \bar{q}_{41} & \bar{q}_{42} & \bar{q}_{43} & \bar{q}_{44} \end{bmatrix} \quad (50)$$

$$\bar{R} = \left[\bar{r}_{11}\right] \quad (51)$$

where $\bar{q}_{11}=X_1$, $\bar{q}_{12}=\bar{q}_{21}=X_2$, $\bar{q}_{13}=\bar{q}_{31}=X_3$, $\bar{q}_{14}=\bar{q}_{41}=X_4$, $\bar{q}_{22}=X_5$, $\bar{q}_{23}=\bar{q}_{32}=X_6$, $\bar{q}_{24}=\bar{q}_{42}=X_7$, $\bar{q}_{33}=X_8$, $\bar{q}_{34}=\bar{q}_{43}=X_9$, $\bar{q}_{44}=X_{10}$ and $\bar{r}_{11}=X_{11}$. In a consequence, we have 11 parameters $(X_1, X_2, \ldots, X_{11})$ to be encoded in the ABC algorithm as a solution vector. Hence, the number of optimization parameters $D_{op}=11$.

In the ABC algorithm training, the linear model of the inverted pendulum with DC motor given in Equations 34-37 is used. Parameters of the DC motor and parameters of the inverted pendulum are presented in Table 2 and Table 3, respectively.

Since the state-space representation of a continuous time LTI control system can be written as in Equation 38, the matrices A, B and C can be defined as follows

$$A = \begin{bmatrix} 0 & 1.0000 & 0 & 0 \\ 0 & -9.0898 & 0.6893 & -0.0424 \\ 0 & 0 & 0 & 1.0000 \\ 0 & -18.9542 & 21.8933 & -1.3477 \end{bmatrix},$$
$$B = \begin{bmatrix} 0 \\ 5.0787 \\ 0 \\ 10.5901 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \tag{52}$$

Table 2. Parameters of the DC motor [4]

Parameter	Value
$\rho(m)$	0.0314
$R(\Omega)$	2.5
K_t	0.05
K_b	0.05
n_1	18.84
n_2	0.986

Table 3. Parameters of the inverted pendulum [4]

Parameter	Unit	Value
M	kg	2.4
m	kg	0.23
l	m	0.36
g	m/s^2	9.81
J_p	kgm^2	0.0099
d	Nms/rad	0.05
b	Ns/m	0.00005

The linearized state space representation of inverted pendulum is used to calculate K and $\bar{N}$. Also the linear model of the pendulum is used to simulate the inverted pendulum and calculate the objective function given in Equation 46. In every cycle of the ABC algorithm, new Q and R matrices are updated and a new feedback gain matrix K is obtained in the following way:

- Solving the algebraic Ricatti equation given in Equation 42 for P where A and B are given above. Q and R are updated by the ABC algorithm.
- Finding the gain matrix using Equation 41.

Also $\bar{N}$ is obtained from Equation 48. After the calculation of K and $\bar{N}$, results are simulated by using the linear inverted pendulum model as shown in Figure 2 and the objective function given in Equation 47 is calculated based on the simulation results. The weighting matrices which have provided the best fitness are memorised. The weighting matrices determined by the ABC algorithm are

$$Q = \begin{bmatrix} 625.8682 & 2.6121 & 2.6121 & 23.1362 \\ 2.6121 & 0.0400 & 0.0400 & 0.1221 \\ 2.6121 & 0.0400 & 0.0400 & 0.1221 \\ 23.1362 & 0.1221 & 0.1221 & 0.8782 \end{bmatrix} \tag{53}$$
$$R = \begin{bmatrix} 13.8236 \end{bmatrix}$$

7. COMPARISON RESULTS

After the LQR training which is performed to determine the matrices Q and R using the ABC algorithm is achieved, the performance of the proposed method is compared with a method proposed by Ghosh et al. [10]. Block diagram of the LQR controller of the nonlinear inverted pendulum model with DC motor shown in Figure 3 is employed for the simulation test. State equations described in Equations 30 – 33 are used to simulate the nonlinear system. These equations are solved using the fourth-order Runge-Kutta method in a numerical manner where the step size $h=0.001$.

To obtain feedback gain matrix K, the algebraic Ricatti equation given in Equation 42 is solved for P where A and B are given in Equation 52, Q and R are given in Equation 53. Then by using the Equation 41, the feedback gain matrix is obtained as

$$K = [-6.7287 \quad -7.6433 \quad 24.6789 \quad 4.7350] \quad (54)$$

Using Equation 48 and based on K and matrices A, B and C, scale factor can be determined as

$$\bar{N} = -6.7287 \quad (55)$$

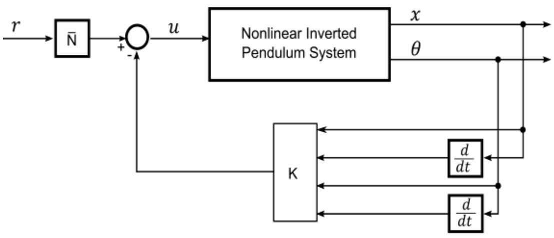


Figure 3. Block diagram of the LQR controller for the nonlinear inverted pendulum with DC motor

The goal of this simulation is to show the changes on position of the cart x, pendulum angle θ and control signal u based on initial conditions of $[x_0 \ \dot{x}_0 \ \theta \ \dot{\theta}_0]$ and reference signal r. Comparing the proposed method with another method will provide a clearer view about the performance. Some of the methods proposed in the literature are mentioned before. It is not required to compare the proposed method with one trained by another swarm optimization algorithm since the main goal of this study is to show advantage of the ABC algorithm in comparison with any method based on trial and error instead of its training performance. The method proposed by Ghosh et al. [10] is used for comparison purposes in this study. According to this method Q and R matrices can be defined as follows [10]:

$$Q = diag\{q_{11}, q_{22}, q_{33}, q_{44}\}, R = r_{11} \quad (56)$$

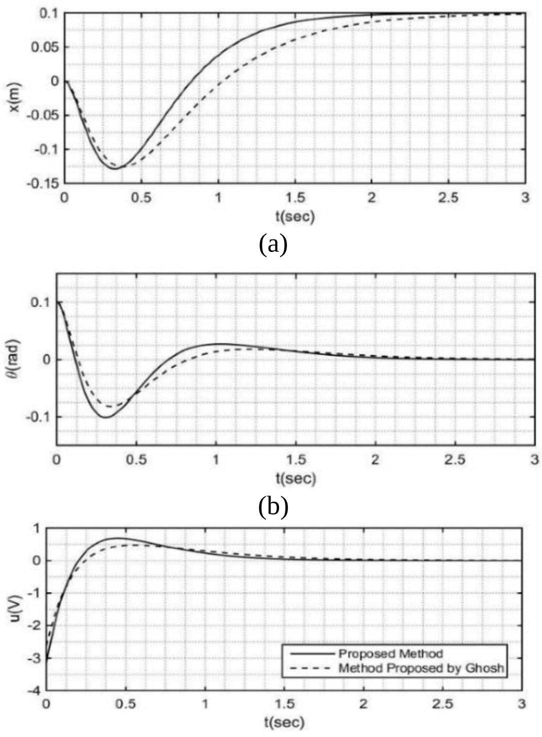
where $q_{11}=500q$, $q_{22}=20q$, $q_{33}=20q$, $q_{44}=q$ and $r_{11}=10^n$. By trial and error and with the choices $q=4$ and $n=2$, Q and R matrices are obtained as follows: $Q=diag\{2000,80,80,4\}$ and $R=[100]$.

To obtain feedback gain matrix for this method, the algebraic Ricatti equation given in Equation 42 is solved for P where A and B are given in Equation 52, Q and R are given above. Then using the Equation 41, the feedback gain matrix is obtained as

$$K_c = [-4.4721 \quad -6.4778 \quad 21.7808 \quad 4.1618] \quad (57)$$

Using Equation 48 and based on K_c and matrices A, B and C, scale factor can be determined as

$$\overline{N_c} = -4.4721 \quad (58)$$



(a)

(b)

(c)

Figure 4. Comparison results of position (a), angle (b) and control signal (c) for $r = 0.1$ and initial conditions $[x_0 \ \dot{x}_0 \ \theta_0 \ \dot{\theta}_0]$=[0 0 0.1 0]

The proposed method and the method proposed in [10] are simulated in the simulation setup which is shown in Figure 3. The only differences between these simulations are the feedback gain matrix K and the scale factor $\bar{N}$. For the proposed method the feedback matrix K= [-6.7287-7.6433 24.6789 4.7350] and $\bar{N}$=-*6.7287* are defined in Equation 54 and Equation 55, respectively, and also for the method proposed in [10] the feedback matrix K_c=[-4.4721-6.4778 21.7808 4.1618] and $\bar{N}$=-4.4721 are defined in Equation 57 and Equation 58, respectively. For both methods performance results are presented in Table 4 where initial conditions $[x_0\ \dot{x}_0\ \theta_0\ \dot{\theta}_0]=[0\ 0\ 0.1\ 0]$ and reference signal $r = 0.1$. Also simulation time and step size are defined as $T = 3$ sec and $h = 0.001$, respectively. Cart position, pendulum angle and control signal are plotted in Figure 4(a-c), respectively. The proposed method was indicated by solid line and the other method was indicated by dotted line in these figures. The proposed method is faster about 0.8sec than the other method to bring the cart to desired position as shown in Figure 4(a). The cart reached the desired position in about 2 sec with the proposed method while it took 2.8 sec in the other method. Pendulum's swinging bound for the proposed method is a bit larger than the other method as shown in Figure 4(b), despite the proposed method managed to keep the pendulum upright position in about 2.2 sec while it took about 2.7 sec for the other method. Overall, the simulation results have shown that the proposed method is more efficient than the method proposed in [10] to select weighting matrices for LQR controller design.

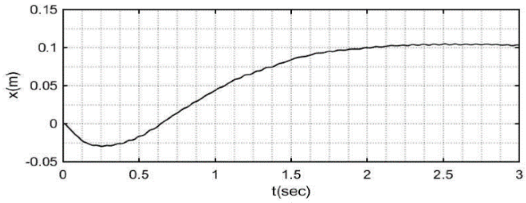
Table 4. Comparison results

	Proposed Method	Method Proposed in [10]
$t_s(sec)$	2.0419	2.8010
$os(\%)$	0	0
e_{ss}	0.0001	0.0014
f_{sum}	2.0420	2.8024

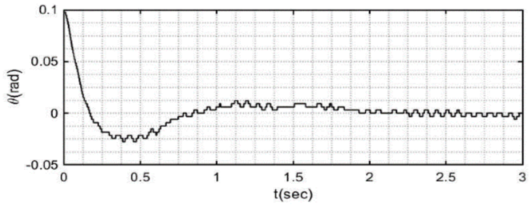
8. EXPERIMENTAL RESULTS

After the comparison, four different experiments are carried out to investigate the performance of the weighting matrices in a real system.

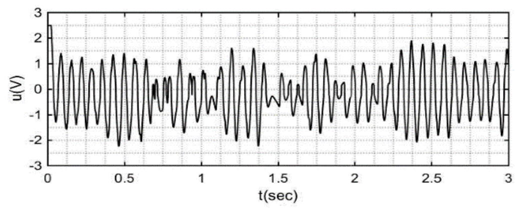
Experiments have performed on Feedback's 33-200 digital pendulum mechanical unit which consists of a cart driven inverted pendulum and a belt with DC motor on adjustable feet. The PC with PCI 1711 Advantech card serves as the main control unit. The control signal, which is a voltage between –2.5 V and 2.5 V, is transferred to the Digital Pendulum Controller (DPC), which drives the DC motor. The cart position and the pendulum angle encoder signals are transferred to the (DPC) and then to the PC.



(a)



(b)

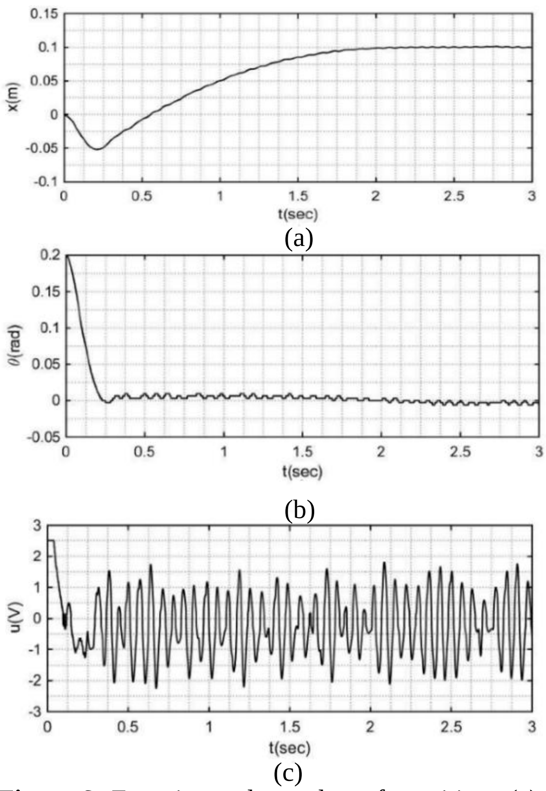


(c)

Figure 5. Experimental results of position (a), angle (b) and control signal (c) for r=0.1 and initial conditions $[x_0\ \dot{x}_0\ \theta_0\ \dot{\theta}_0]=[0\ 0\ 0.1\ 0]$

The goals of these experimental tests are to show the changes on position of the cart x, pendulum angle θ and control signal u based on initial conditions of $[x_0\ \dot{x}_0\ \theta_0\ \dot{\theta}_0]$ and reference signal r on the experimental setup.

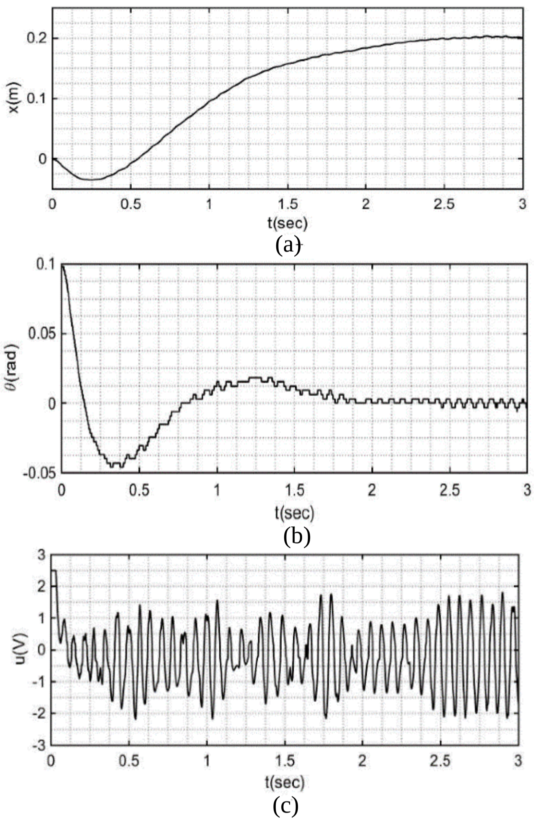
In the first test, for reference signal $r = 0.1$ and initial conditions $[x_0\ \dot{x}_0\ \theta_0\ \dot{\theta}_0]$=[*0 0 0.1 0*], the cart position (x) , pendulum angle (θ) and control signal (u) are plotted and shown in Figure 5(a-c), respectively. In the first test, the controller managed to bring the cart from the initial position $0m$ to the desired position in about 2 sec while bringing the pendulum angle from 0.1 rad to 0 rad as shown in Figure 5(a). Also the pendulum angle reached to the desired position from the initial position in about 2 sec as shown in Figure 5(b). The control signal started from 2.5 V and during the test it was in the range of −2.5 V and +2.5 V so it obeys physical restriction as shown in Figure 5(c).



(a) (b) (c)

Figure 6. Experimental results of position (a), angle (b) and control signal (c) for $r = 0.1$ and initial conditions $[x_0\ \dot{x}_0\ \theta_0\ \dot{\theta}_0]$=[*0 0 0.2 0*]

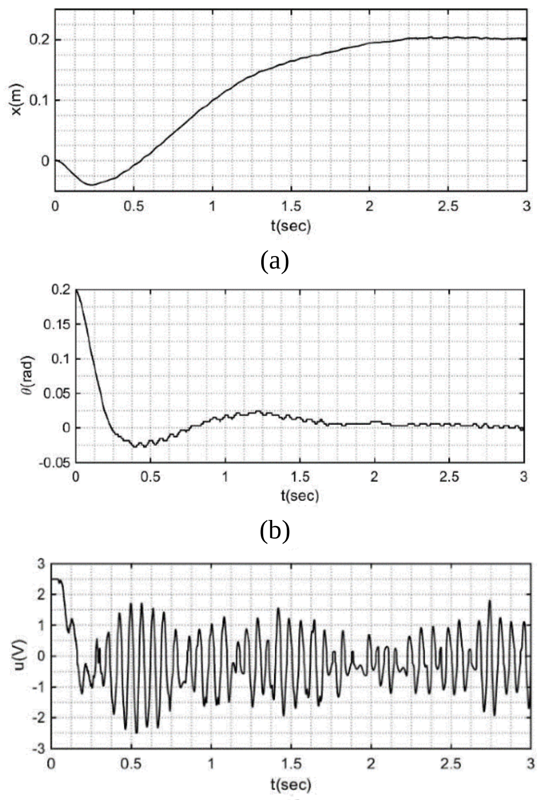
In the second test, for reference signal r = 0.1 and initial conditions $[x_0\ \dot{x}_0\ \theta_0\ \dot{\theta}_0]$=[*0 0 0.2 0*], the cart position (x), pendulum angle (θ) and control signal (u) are plotted and shown in Figure 6(a-c), respectively. In the second test, the controller managed to bring the cart from the initial position 0 m to the desired position in about 2 sec as shown in Figure 6(a). The pendulum reached to the upright position from the initial position in about 0.4 sec as shown in Figure 6(b). The control signal started from 2.5 V and during the test it was in the range of −2.5 V and +2.5 V as shown in Figure 6(c).



(a) (b) (c)

Figure 7. Experimental results of position (a), angle (b) and control signal (c) for $r = 0.2$ and initial conditions $[x_0\ \dot{x}_0\ \theta_0\ \dot{\theta}_0]$=[*0 0 0.1 0*]

In the third test, for reference signal $r = 0.2$ and initial conditions $[x_0\ \dot{x}_0\ \theta_0\ \dot{\theta}_0]$=[*0 0 0.1 0*], the cart position (x), pendulum angle (θ) and control signal (u) are plotted and shown in Figure 7(a-c), respectively. In the third test, the controller managed to bring the cart from the initial position 0 m to the desired position in about 2.4 sec while bringing the pendulum angle from 0.1 rad to 0 rad as shown in Figure 7(a). Also the pendulum angle reached to the desired position from the initial position in about 2 sec as shown in Figure 7(b). The control signal started from 2.5 V and during the test it was in the range of -2.5 V and $+2.5 \text{ V}$ as shown in Figure 7(c).



(a)

(b)

(c)

Figure 8. Experimental results of position (a), angle (b) and control signal (c) for $r = 0.2$ and initial conditions $[x_0 \; \dot{x}_0 \; \theta_0 \; \dot{\theta}_0]$=[*0 0 0.2 0*]

In the fourth test, for reference signal $r = 0.2$ and initial conditions $[x_0 \; \dot{x}_0 \; \theta_0 \; \dot{\theta}_0]$=[*0 0 0.2 0*], the cart position (x), pendulum angle (θ) and control signal (u) are plotted and shown in Figure 8(a-c), respectively. In the fourth test, the controller managed to bring the cart from the initial position 0 m to the desired position in about 2.3 sec as shown in Figure 8(a). Also the pendulum angle reached to the desired position from the initial position in about 2.2 sec as shown in Figure 8(b). The control signal started from 2.5 V and during the test it was in the range of -2.5 V and $+2.5 \text{ V}$ as shown in Figure 8(c).

9. CONCLUSION

The current study has presented a new ABC algorithm based strategy that focuses on efficiency of LQR controller design process. The ABC algorithm has been employed to determine LQR controller weighting matrices and an LQR controller has been designed for an inverted pendulum system with DC motor. The designed LQR controller was tested with different reference signals and different initial conditions. It is well known that the ABC algorithm has good performances in solving numerical optimization problems. Thus, the effectiveness of the LQR controller design using the ABC algorithm was researched and a satisfactory performance was obtained. The comparison results have shown that using the ABC algorithm is more effective and feasible to select weighting matrices for LQR controller design than the method proposed by Ghosh et al. [10]. Also it has shown that the proposed method can optimize multiple time domain control system specifications such as settling time, overshoot and steady state error, simultaneously.

There may exist some varying parameters, disturbances, and uncertainties in the real-life plants. Therefore, the mathematical model of the system is very important. Experimental results have shown that the inverted pendulum model with DC motor used for controller design is quite realistic. Also the experimental results have shown that the proposed method can be used in real-life plants efficiently. The proposed method is flexible and applicable in a wide range of optimization problems. Hence, it can be regarded as a general controller design method that can be applied to a wide class of control problems.

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